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Unsteady heat transfer from a suddenly heated infinite plate placed in a rarefied gas

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Abstract. Heat transfer from a plate placed in a rarefied gas and suddenly heated is investigated from the point of view of the kinetic theory of gases. A model kinetic equation is solved using the method of moments. Analytical formulae for the density and temperature jumps at the surface of the plate are obtained. It is found that in the free molecular regime the temperature jump is constant while in the nearly-free molecular regime it is a linear function of time.

1. Introduction

One of the problems of great interest in aerodynamics is the investigation of the behaviour of a rarefied gas near a suddenly heated body. It is important to determine the rule governing heat transfer due to collisions of the molecules with solid surfaces and collisions between the molecules themselves. Because of rarefaction of the gas there must be discontinuities in the macroscopic parameters at the solid surface. The jumps in these quantities at the surface due to the discontinuities must be determined. The present paper deals with heat transfer from a suddenly heated plate placed in a highly rarefied gas; our aim is mainly to determine the density and temperature jumps at the surface of the plate. These jumps can be obtained if the density and temperature distributions at any instant and at any point are obtained.

The distribution function is assumed to satisfy the Boltzmann equation and its initial form is taken to be Maxwellian.

Reflection of molecules from the surface is considered completely diffuse, this means that the distribution of the molecules reflected from the surface is Maxwellian and is dependent on temperature (i.e. the temperature of the plate). At infinity, the density and temperature are considered bounded. The collision term is simplified by using the model suggested simultaneously by Bhatnagar *et al* (1954) and Wellander (1954). The technique used to solve the kinetic equation is the method of moments with a two-sided distribution function. This method assumes that at any point in the velocity space the distribution function is discontinuous. There are two types of molecules: those reflected from the solid surface that do not suffer collisions with other molecules and those which are not reflected by the plate; these types have different distribution functions. The expression for the distribution function contains some unknowns which are determined from the moment equation. In general this method can be used to solve non-linear problems. It was proved by Khadr (1970) that this method gives better results than other five available methods when solving the problem of Couette flow. Also Kashmarov (1963) uses it to solve the linear problem of shear stress. Kashmarov considers the motion of the gas near a plate with constant density and temperature which is suddenly moved, he uses the method of moments with a two-sided distribution function to get a solution suitable for any density and his results agree with the results of other investigators.

2. Basic equations

We consider a semi-infinite space, which is bounded by a very long (nearly infinite) plate and filled with a fixed rarefied gas. We consider the unsteady heat transfer from the infinite plate when it is suddenly heated. It is well known that the distribution function satisfies the Boltzmann kinetic equation. In this case the equation has the form:

$$\frac{\partial F}{\partial t}+C_y\frac{\partial F}{\partial y}=\Delta_c F \tag{1}$$

where $F=F(\bar{C}, y, t)$ is the distribution function, $\bar{C}$ is the velocity of molecules and $\Delta_c F$ is the collision term of the Boltzmann equation. To solve (1) we shall use the method of moments with discontinuous distribution function (Shidlovskiy 1967) as:

$$F=\begin{cases} F_1=\dfrac{n_1}{(2\pi RT_1)^{3/2}}\exp\left(-\dfrac{C^2}{2RT_1}\right) & C_y<0 \\ F_2=\dfrac{n_2}{(2\pi RT_2)^{3/2}}\exp\left(-\dfrac{C^2}{2RT_2}\right) & C_y>0 \end{cases} \tag{2}$$

where n_1, n_2, T_1 and T_2 are unknown functions of the variables y and t (it is assumed that the plate is situated in the plane $y=0$). The moments equations are obtained by multiplying equation (1) by some dynamical variable $\phi_i(\bar{C})$ and integrating over $\bar{C}$ from $-\infty$ to $-\infty$; then we have:

$$\frac{\partial}{\partial t}\int \phi_i F\,\mathrm{d}\bar{C}+\frac{\partial}{\partial y}\int C_y\phi_i F\,\mathrm{d}\bar{C}=\int(\Delta_c F)\phi_i\,\mathrm{d}\bar{C} \tag{3}$$

where the integrals over the velocity space are evaluated from:

$$\int g(\bar{C})F\,\mathrm{d}\bar{C}=\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} gF_1\,\mathrm{d}\bar{C}+\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} gF_2\,\mathrm{d}\bar{C}. \tag{4}$$

To determine the four unknowns n_1, n_2, T_1 and T_2 we choose four linearly independent functions ϕ_i as:

$$\phi_1=1, \qquad \phi_2=C_y, \qquad \phi_3=C^2, \qquad \phi_4=\tfrac{1}{2}C^2C_y. \tag{5}$$

The calculation of the right-hand side of equation (3) is appreciably simplified by approximating the collision term in equation (1) in the form (Bhatnager *et al* 1954, Wellander 1954)

$$\Delta_c F=-\frac{nRT}{\mu}(F^{(0)}-F) \tag{6}$$

where n is the density, R is the gas constant, T is the temperature and μ is the viscosity coefficient. This corresponds to the first approximation of the Chapman–

Enskog theory (Shidlovskiy 1967) and for the interaction model $F = K/d^5$ (d being the intermolecular distance and K a constant), $\mu = 2kT(m/2K)^{1/2}/3A_2(5)$ (k is Boltzmann's constant, $A_2(5) = 1{\cdot}3682$).

We now define dimensionless variables $\bar{n}$, $\bar{T}$, $\bar{y}$, $\bar{t}$ as:

$$n_i = n_0\bar{n}_i, \qquad T_i = T_0\bar{T}_i, \qquad y = \bar{y}l, \qquad t = \bar{t}l/(2RT_0)^{1/2}$$

where n_0 is the initial density, T_0 is the temperature of the gas and l the length of the plate. Dropping the bars on the dimensionless quantities we get:

$$\frac{\partial}{\partial t}(n_1+n_2)+\frac{1}{\pi^{1/2}}\frac{\partial}{\partial y}(n_2T_2^{1/2}-n_1T_1^{1/2})=0 \tag{7}$$

$$\frac{\partial}{\partial t}(n_2T_2^{1/2}-n_1T_1^{1/2})+\frac{\pi^{1/2}}{2}\frac{\partial}{\partial y}(n_1T_1+n_2T_n)=0 \tag{8}$$

$$\frac{\partial}{\partial t}(n_1T_1+n_2T_2)+\frac{4}{3\pi^{1/2}}\frac{\partial}{\partial y}(n_2T_2^{3/2}-n_1T_1^{3/2})=0 \tag{9}$$

$$\begin{aligned}\frac{\partial}{\partial t}&(n_2T_2^{3/2}-n_1T_1^{3/2})+\frac{5\pi^{1/2}}{8}\frac{\partial}{\partial y}(n_1T_1^2+n_2T_2^2)\\ &=-\delta[\tfrac{2}{3}(n_1+n_2)(n_2T_2^{3/2}-n_1T_1^{3/2})+\tfrac{1}{2}(n_2T_2^{1/2}-n_1T_1^{1/2})(n_1T_1+n_2T_2)],\\ &\delta=\tfrac{3}{4}ln_0A_2(5)(K/kT_0)^{1/2}.\end{aligned} \tag{10}$$

If we consider the initial distribution function to be Maxwellian, then

$$n_1 = n_2 = 1, \qquad T_1 = T_2 = 1 \qquad \text{at } t = 0 \tag{11}$$

and the boundary condition for the normal velocity, $nV_y = \int C_yF\,\mathrm{d}\bar{C} = 0$, gives

$$[n_2T_2^{1/2}-n_1T_1^{1/2}]_{y=0}=0. \tag{12}$$

The condition that the distribution function of the reflected molecule be Maxwellian dependent on temperature (equal to the temperature of the plate) gives:

$$T_2(0,t)=1+\chi' \tag{13}$$

where the plate is assumed heated to the temperature $T_0(1+\chi')$. Also

$$n_1, n_2, T_1 \text{ and } T_2 \text{ are considered bounded at } y=\infty. \tag{14}$$

To solve equations (7)–(10), we shall use the small-parameter method; considering χ' as the small parameter, and neglecting all terms of order $O(\chi'^2)$, we put

$$\begin{aligned}&n_1 = 1+\chi'n_i', \qquad T_i = 1+\chi'T_i' \qquad (i=1,2)\\ &X = n_1'+n_2', \qquad Y = n_2'-n_1', \qquad Z = T_1'+T_2', \qquad L = T_2'-T_1'.\end{aligned} \tag{15}$$

Substituting from (15) into (7)–(10) and equating the coefficients of χ' on both sides of equations (7)–(10) we get the following partial differential equations:

$$\frac{\partial X}{\partial t}+\frac{1}{\pi^{1/2}}\frac{\partial Y}{\partial y}+\frac{1}{2\pi^{1/2}}\frac{\partial L}{\partial y}=0 \tag{16}$$

$$\frac{\partial Y}{\partial t}+\frac{1}{2}\frac{\partial L}{\partial y}+\frac{\pi^{1/2}}{2}\frac{\partial X}{\partial y}+\frac{\pi^{1/2}}{2}\frac{\partial Z}{\partial y}=0 \tag{17}$$

$$\frac{\partial Z}{\partial t}+\frac{1}{3\pi^{1/2}}\frac{\partial Y}{\partial y}+\frac{3}{2\pi^{1/2}}\frac{\partial L}{\partial y}=0 \tag{18}$$

$$\frac{\partial L}{\partial t}+\frac{\pi^{1/2}}{8}\frac{\partial X}{\partial y}+\frac{3\pi^{1/2}}{4}\frac{\partial Z}{\partial y}=-\delta(\tfrac{7}{3}Y+\tfrac{5}{2}L). \tag{19}$$

The initial condition (11) becomes:

$$X=Y=Z=L=0 \qquad \text{at } t=0. \tag{20}$$

Also the boundary conditions (12), (13) are:

$$Y(0,t)+\tfrac{1}{2}L(0,t)=0 \tag{21}$$

$$Z(0,t)+L(0,t)=2 \tag{22}$$

$$X,\ Y,\ Z,\ L \text{ are bounded at } y=\infty. \tag{23}$$

3. The solution in the case of highly rarefied system

In this case we can neglect the collisions between the particles, i.e. take $\delta=0$ in (16)–(19). Using Laplace's transform, the solutions of equations (16)–(19) are Osman (1976)

$$X^{(0)}=a_1H(t-ay)+a_2H(t-a'y) \tag{24}$$

$$Y^{(0)}=b_1H(t-ay)+b_2H(t-a'y) \tag{25}$$

$$Z^{(0)}=c_1H(t-ay)+c_2H(t-a'y) \tag{26}$$

$$L^{(0)}=d_1H(t-ay)+d_2H(t-a'y) \tag{27}$$

where the constants:

$a=1\cdot8069$,	$a'=0\cdot8573$,	$a_1=-0\cdot7259$,	$a_2=0\cdot3446$,
$b_1=-0\cdot9046$,	$c_1=0\cdot2812$,	$d_1=0\cdot3849$,	
$b_2=0\cdot34417$,	$c_2=0\cdot5928$,	$d_2=0\cdot7410$	

and

$$H(t-ay)=\begin{cases}1 & t>ay\\ 0 & t\leqslant ay.\end{cases} \tag{28}$$

To solve equations (16)–(19) when the collisions are taken into account, while the collision term may be taken small, we use the small-parameter method. The quantities X, Y, Z and L are expressed as:

$$X=X^{(0)}+\delta X^{(1)}, \qquad Y=Y^{(0)}+\delta Y^{(1)},$$
$$Z=Z^{(0)}+\delta Z^{(1)}, \qquad L=L^{(0)}+\delta L^{(1)}. \tag{29}$$

The initial conditions for $X^{(0)}$, $Y^{(0)}$, $Z^{(0)}$ and $L^{(0)}$ are

$$X^{(0)}(y,0)=Y^{(0)}(y,0)=Z^{(0)}(y,0)=L^{(0)}(y,0)=0$$

and the boundary conditions are

$$Y^{(0)}(0,t)+\tfrac{1}{2}L^{(0)}(0,t)=0$$
$$Z^{(0)}(0,t)+L^{(0)}(0,t)=2$$
$$X^{(0)}, Y^{(0)}, Z^{(0)}, L^{(0)} \text{ are bounded at } y=\infty.$$

Also the initial and boundary conditions for $X^{(1)}$, $Y^{(1)}$, $Z^{(1)}$ and $L^{(1)}$ are

$$X^{(1)}(y,0)=Y^{(1)}(y,0)=Z^{(1)}(y,0)=L^{(1)}(y,0)=0$$
$$Y^{(1)}(0,t)+\tfrac{1}{2}L^{(1)}(0,t)=0$$
$$Z^{(1)}(0,t)+L^{(1)}(0,t)=0$$
$$X^{(1)}, Y^{(1)}, Z^{(1)}, L^{(1)} \text{ are bounded at } y=\infty.$$

Substituting from (29) in (16)–(19) and equating the coefficients of δ on both sides of these equations we get partial differential equations for $X^{(1)}$, $Y^{(1)}$, $Z^{(1)}$, $L^{(1)}$. Using Laplace's transform we can get the solution in the form (Osman 1976)

$$X^{(1)}=(K_1t+R_1y)H(t-ay)+(M_1t+N_1y)H(t-a'y) \tag{30}$$
$$Y^{(1)}=(K_2t+R_2y)H(t-ay)+(M_2t+N_2y)H(t-a'y) \tag{31}$$
$$Z^{(1)}=(K_3t+R_3y)H(t-ay)+(M_3t+N_3y)H(t-a'y) \tag{32}$$
$$L^{(1)}=(K_4t+R_4y)H(t-ay)+(M_4t+N_4y)H(t-a'y) \tag{33}$$

where

$K_1=-1\cdot8048$,	$M_1=1\cdot8047$,	$R_1=2\cdot4294$,	$N_1=-1\cdot0740$,
$K_2=-2\cdot3038$,	$M_2=1\cdot9407$,	$R_2=3\cdot1263$,	$N_2=-1\cdot1946$,
$K_3=1\cdot1544=-M_3$		$R_3=-1\cdot7637$,	$N_3=1\cdot8037$,
$K_4=1\cdot5359=-M_4$		$R_4=-2\cdot3342=N_4$.	

5. Discussion of results and conclusions

The dimensionless average density and temperature are given by

$$n=\tfrac{1}{2}(n_1+n_2) \tag{34}$$
$$T=(n_1T_1+n_2T_2)/(n_1+n_2). \tag{35}$$

(1) In the case of highly rarefied gas we get

$$n/n_0=1+\tfrac{1}{2}\chi'(a_1H(t-ay)+a_2H(t-a'y))$$
$$T/T_0=1+\chi'(a_1'H(t-ay)+a_2'H(t-a'y))$$
$$a_1'=-0\cdot5853, \qquad a_2'=0\cdot6410.$$

(2) At the surface of the plate ($y=0$) we get

$$n(0,t)/n_0=1+\tfrac{1}{2}\chi'(a_1+a_2) \qquad \text{for } t>0$$
$$T(0,t)/T_0=1+\chi'(a_1'+a_2') \qquad \text{for } t>0.$$

Then the dimensionless temperature jump T_j (the difference between gas and plate temperatures) at the surface of the plate will be:

$$T_i(0, t)/T_0 = T_p - T_g = 1 + \chi' - T(0, t) = 0{\cdot}9443\chi' \qquad \text{for } t > 0$$

where T_p is the temperature of the plate and T_g is the temperature of the gas.

We conclude that the temperature jump at the surface of the plate at any instant is constant if collisions between molecules are neglected.

(3) In the case when the collisions are considered we get:

$$n(0, t)/n_0 = 1 + \tfrac{1}{2}\chi'[(a_1 + K_1\delta t + R_1\delta y)H(t - ay) + (a_2 + M_1\delta t + N_1\delta y)H(t - a'y)]$$

$$T(0, t)/T_0 = 1 + \chi'[(A + B\delta t + D\delta y)H(t - ay) + (A' + B'\delta t + D'\delta y)H(t - a'y)]$$

where

$$A = -0{\cdot}5853, \qquad B = -1{\cdot}2276, \qquad D = -1{\cdot}5475,$$

$$A' = 0{\cdot}6410, \qquad B' = 1{\cdot}2275, \qquad D' = -0{\cdot}1721.$$

(4) At the surface of the plate ($y = 0$), there is a temperature jump T_j at any instant given by

$$T_j(0, t) = T_p - T_g = (0{\cdot}9443 + 0{\cdot}0001\delta t).$$

To conclude we remark that the above results are obtained for an infinite plate. An interesting problem would be to consider the situation for a plate only partially bounding the gas. In this case the solution will contain an accommodation coefficient. The analytical solution may be compared with experimental results, which could be obtained. From that comparison the behaviour of the accomodation coefficient could be investigated. The thermal waves propagated in the gas have a special form and we think that these waves may be useful in engineering applications.

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